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ON THE EFFICIENCY OF SENSITIVITY EXPERIMENTS ANALYZED BY THE MAXIMUM LIKELIHOOD ESTIMATION PROCEDURE UNDER THE CUMULATIVE NORMAL RESPONSE

Kali S. Banerjee

September 1980

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
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I. INTRODUCTION

The problem of analysis of data coming from "sensitivity experiments" in the context of Ballistics Research is classically old. Equally classical is the problem of analysis of data originating from the statistically analogous and the conceptually parallel experiments conducted in biological and pharmaceutical research in dosage mortality estimations. Such estimation procedure has even a wider horizon, as the same methodology is applied also to some reliability studies insofar as such studies are characterized by the "stress-strength" phenomenon. In all these areas, there has indeed been a considerable accumulation of data which have been analyzed from time to time by various methods depending upon the types and designs of the experiments suited to different models characterizing the distribution of the response function. However, one of the methods, which is frequently resorted to, is the use of the maximum likelihood estimation procedure in the set-up of the cumulative normal distribution as the model for describing the probability of "success" (or "failure") varying with the intensity of the stimulus applied. It is generally believed that this procedure of estimation in the framework of normal response is still one of the best available so far. Hence, in response to frequent requests for analysis of such data, the BRL of Aberdeen Proving Ground has coded the solutions (see Golub and Grubbs[1]) providing the estimates of μ and σ along with the estimates of their asymptotic variances.

II. THE AIM OF THE PRESENT PAPER

The estimates of the asymptotic variances referred to above would no doubt provide, in themselves, some measure of efficiency of the estimates of μ and σ; but, it appears that no measure has been made available yet that would show how efficient these estimates are, and to what extent the efficiency could be further improved. There is also a need for a basis to measure the relative efficiency of the experiment as a whole, the calculation of which would be based upon the maximum possible precision attainable under such an estimation procedure. Such a measure, if it could be provided, would furnish an idea of the relative efficiencies of similar experiments conducted at different times, pointing possibly to the necessity for re-assessment in some cases. The aim of this paper is to provide such a measure of efficiency. The calculation of efficiency as presented here is based on the exposition of some simple statistics. Such an exposition is found to be helpful in understanding more visibly some of the well known implications and limitations of the maximum likelihood estimation procedure in the given context.

[1]Golub, A., and Grubbs, F.E., "Analysis of sensitivity experiments when the levels of stimulus cannot be controlled," Ann. Math. Stat. 57 (1956), 257-265.

In indicating how the maximum possible precision can be attained, we also aim at suggesting a good design for such an experiment in the situation where the levels of the stimulus could be reasonably controlled.

It is not known to the present authors if anybody else has, in the past, attempted such an analysis based upon the exposition of the simple statistics as presented in this paper.

III. MAXIMUM LIKELIHOOD ESTIMATION PROCEDURE

A. Estimating Equations

It is assumed that the probability p_i of success with the stimulus x_i is of the form,

$$p_i = \int_{-\infty}^{t_i} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt, \quad t_i = \frac{x_i - \mu}{\sigma}, \qquad (1)$$

where μ and σ are the unknown parameters which require to be estimated.

We then let Y_i to be a random variable taking the value 1 or 0, depending upon whether the outcome of the trial is a success or a failure at the level x_i with the probability of success as p_i = Prob $(Y_i = 1)$, and the probability of failure as $q_i = 1 - p_i$ = Prob $(Y_i = 0)$.

It may be noted here that success and failure are relative terms and that we can interchange the definitions. In this case and for the purpose of discussion in this paper, we assume that the probability of success becomes larger with the increased level of the stimulus.

The likelihood equation is given by

$$L = \prod_{i=1}^{n} p_i^{y_i} (1-p_i)^{1-y_i} , \qquad (2)$$

where p_i, as noted in (1), is a function of μ and σ, and the estimating equations are obtained as;

$$\frac{\partial \log L}{\partial \mu} = \frac{1}{\sigma} \sum_{i=1}^{n} \left[\frac{z_i(p_i - y_i)}{p_i(1-p_i)} \right] = 0 \quad , \tag{3.1}$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{1}{\sigma} \sum_{i=1}^{n} \left[\frac{z_i t_i (p_i - y_i)}{p_i(1-p_i)} \right] = 0, \tag{3.2}$$

where
$$z_i = \frac{1}{\sqrt{2\pi}} e^{-t_i^2/2}.$$

For a given set of data, we may actually substitute the values of y_i as 1 or 0, depending upon whether the given stimulus has brought a success or a failure, and obtain the equations in the following forms,

$$\frac{1}{\sigma} \left[\sum_f \frac{z_f}{q_f} - \sum_s \frac{z_s}{p_s} \right] = 0 \tag{4.1}$$

$$\frac{1}{\sigma} \left[\sum_f \frac{t_f z_f}{q_f} - \sum_s \frac{t_s z_s}{p_s} \right] = 0, \tag{4.2}$$

where the subscripts s and f refer respectively to success and failure.

It may be pointed out here that z_i, p_i, q_i are all functions of t_i. The argument t_i is omitted here for the sake of notational simplicity. It will be introduced whenever necessary to show the argument.

Since the equations are not directly soluble, an iteration scheme as embodied in the Newton-Raphson procedure is used to provide the estimates of μ and σ. For the details of calculation, one may refer, among others, to Golub and Grubbs.[1]

B. Variances of the Estimates

The asymptotic variance-covariance matrix (see Golub and Grubbs[1]) is given by

$$A^{-1} = \begin{bmatrix} A_{\mu\mu} & A_{\mu\sigma} \\ A_{\mu\sigma} & A_{\sigma\sigma} \end{bmatrix}^{-1} = \begin{bmatrix} A^{\mu\mu} & A^{\mu\sigma} \\ A^{\mu\sigma} & A^{\sigma\sigma} \end{bmatrix} \tag{5}$$

where

$$A = -E \begin{bmatrix} \frac{\partial^2 \log L}{\partial_\mu^2} & \frac{\partial^2 \log L}{\partial_\mu \partial_\sigma} \\ \frac{\partial^2 \log L}{\partial_\mu \partial_\sigma} & \frac{\partial^2 \log L}{\partial_\sigma^2} \end{bmatrix} . \tag{6}$$

In the above, 'E' stands for expectation with reference to Y_i, and $A^{\mu\mu}$, $A^{\mu\sigma}$ and $A^{\sigma\sigma}$ in equation (5) denote, respectively, the variance of $\hat{\mu}$, the covariance between $\hat{\mu}$ and $\hat{\sigma}$, and the variance of $\hat{\sigma}$.

It is easy to show (see, for example, Golub and Grubbs[1]) that;

$$A_{\mu\mu} = \frac{1}{\sigma^2} \sum_i \frac{z_i^2}{p_i q_i} , \tag{7.1}$$

$$A_{\mu\sigma} = \frac{1}{\sigma^2} \sum_i \frac{z_i^2 t_i}{p_i q_i} , \tag{7.2}$$

$$A_{\sigma\sigma} = \frac{1}{\sigma^2} \sum_i \frac{z_i^2 t_i^2}{p_i q_i} . \tag{7.3}$$

IV. LIMITATIONS AND RESTRICTIONS OWING TO SMALL SAMPLE SIZE

The efficiency of the maximum likelihood estimation procedure would depend upon how large is the sample size. A large sample would provide unbiased and minimum variance estimates of μ and σ. Such a claim cannot be made in general when the number of trials is small. Results based on simulation studies from small samples indicate[2] that the estimates of μ are unbiased for all practical purposes, but the estimates of σ are biased, being too small on the average. Variances of $\hat{\sigma}$ have also been found to be rather large. Such an instability of the estimate of σ has also an adverse effect on the efficiency of estimates of μ and σ, as such estimates are proportional to σ^2 (see equations (7.1) - (7.3)).

More importantly, a small sample may not even be amenable to this kind of analysis in some situations. Before the data are subjected to such an analysis, the success-failure sequence has to be examined. In

[2] Visnaw, V. V., and Hagan, J. S., "Analysis of sensitivity data following a normal distribution," Report No. 70-AS-113, Material Testing Directorate, APG, Maryland, 1970.

order that it be possible to estimate both μ and σ with $\sigma \epsilon (0,\infty)$, there should be a zone of "mixed results." If no zone of "mixed results" occurs, no estimate of σ other than 0 can be obtained. (See, for example, Langlie[3]). (No zone of "mixed results" means that the highest stimulus level at which a nonresponse (in this case, a failure) occurred is less than the lowest stimulus level at which a response (in this case, a success) occurred.)

V. ON THE CONSISTENCY OF THE EQUATIONS PROVIDING THE ESTIMATES

A. Introductory Remarks Paving the Background

In consequence of having to deal with a small sample, we may face a situation, as referred to above, where it may not be possible to furnish any maximum likelihood estimate of μ and σ at all, under the assumption that σ is non-zero and finite. In particular, we need a zone of "mixed results," and that we need such a zone is demonstrated in Langlie.[3] As the demonstration in Langlie[3] is based on contrapositive reasoning which is rather involved, one may be interested in seeing more visibly how a zone of "no mixed results" really affects the consistency of the estimating equations.

It has been observed in section III that the equations providing the estimates of μ and σ and the variance-covariance matrix for these estimates depend only upon z/q, z/p and t. Thus, a portrayal of z/q, z/p and the difference, $(z/q-z/p)$, should provide the necessary information required in checking the consistency of the estimating equations.

Diagram I shows the graphs of z/q, z/p and $(z/q - z/p)$ as functions of t. Of these three graphs, we need the first two for the purpose of our immediate discussion. The third graph, that is, the graph of $(z/q - z/p)$ will be referred to in a later section (section VII).

Note that $z(-t) = z(t)$, $p(-t) = q(t)$, and $q(-t) = p(t)$ with $t \epsilon (-\infty,\infty)$. Also, in the positive domain of t with $t \epsilon [0,\infty)$, $z/q \geqslant z/p$, $q \leqslant p$; $z(0)/p(0) = z(0)/q(0) = (2/\pi)^{1/2}$. In the negative domain of t, $z/p \geqslant z/q$, as p takes the place of q.

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[3]Langlie, H. J., "A reliability test method for "one-shot" items," Publication No. U-1792, Reliability Branch, work performed under US Army contract DA-04-495-ORD-1835, 1962.

B. Inevitable Inconsistency with Two Observations

Given that σ is nonzero and finite, two distinct observations would always lead to inconsistency in the estimating equations. Let us suppose that both the observations, given by $t = t_1$ and $t = t_2$, $t_1 < t_2$, are in the positive domain of t, and that one gives a failure and the other, a success. Then, equation (4.1) would reduce to $z(t_1)/q(t_1) = z(t_2)/p(t_2)$. This can never happen when t_1 and t_2 are distinct. If both t_1 and t_2 give successes, then equation (4.1) would reduce to $z_1(t_1)/p(t_1) + z(t_2)/p(t_2) = 0$. Since each of these two terms is positive, and only one can go to 0 as $t \to \infty$, this can never happen. Again, if the success occurs in the positive domain and the failure in the negative domain, then equation (4.1) would reduce to $z(-t_1)/q(-t_1) = z(t_2)/p(t_2)$ which would imply that $z(t_1)/p(t_1) = z(t_2)/p(t_2)$, and this is impossible, when t_1 and t_2 are distinct. However, if success occurs in the negative domain, and failure in the positive domain with $t_1 = -t_2$, then equation (4.1) would be consistent being given by $z(-t_2)/q(-t_2) = z(t_2)/q(t_2) \Rightarrow z(t_2)/q(t_2) = z(t_2)/q(t_2)$. But, equation (4.2) will be inconsistent. Similar inconsistency can be demonstrated in other cases also.

C. Other Types of Inconsistencies

If there are s trials in the positive domain, all of which give successes, equation (4.1) will indicate inconsistency. Alternatively, if there are f trials in the negative domain all of which give failures, equation (4.1) would lead to inconsistency. In each of such cases, there will be no "zone of mixed results."

D. On the Need for a Zone of Mixed Results

Suppose there are three observations in the positive domain given by $t_1 < t_2 < t_3$, and that failure occurs at t_1 and successes at t_2 and t_3. This provides an illustration of a zone of "no mixed results." In this situation, equation (4.1) would reduce to $z(t_1)/q(t_1) = z(t_2)/p(t_2) + z(t_3)/p(t_3)$. It will be evident from an examination of the relative magnitudes of z/q and z/p and their graphs (Diagram I) that such an equation can be consistent, although t_1, t_2 and t_3 cannot take values far to the right. But, equation (4.2) would reduce to $t_1 z(t_1)/q(t_1) = t_2 z(t_2)/p(t_2) + t_3 z(t_3)/p(t_3)$. Such an equation is inconsistent with equation (4.1), since $t_1 z(t_1)/q(t_1) = t_1 z(t_2)/p(t_2) + t_1 z(t_3)/p(t_3) < t_2 z(t_2)/p(t_2) + t_3 z(t_3)/p(t_3)$.

If, on the other hand, we had successes at t_1 and t_3, and failure at t_2, then equation (4.1) would reduce to $z(t_2)/q(t_2) = z(t_1)/p(t_1) + z(t_3)/p(t_3)$. It will be evident from Diagram 1 that such a possibility exists, although not in a zone very far away from 0. Since t_2 lies between t_1 and t_3, equation (4.2) will also be consistent. This illustrates a zone of mixed results.

If t_1 giving a failure lies in the negative domain of t, that is, if t_1 is negative, while t_2 and t_3 are in the positive domain giving success, then equation (4.1) would reduce to $z_1(t_1)/p(t_1) = z(t_2)/p(t_2) + z(t_3)/p(t_3)$, which may not be inconsistent. Equation (4.2) would reduce to $- t_1 z(t_1)/p(t_1) = t_2 z(t_2)/p(t_2) + t_3 z(t_3)/p(t_3)$. This is impossible, since a positive quantity cannot be equal to a negative quantity. This is an illustration of a zone without "mixed results," leading to inconsistency.

We can extend the above reasoning with reference to observations in the negative domain to conclude in general that there will be inconsistency in the equations arising from a zone of "no mixed results."

A scrutiny of the relative magnitudes of z/q and z/p as reflected in the graphical portrayal in Diagram I would reveal another important limitation of this estimation procedure. Observations few in number, but too far away from t = 0, that is, too far away from μ solely in the positive domain of t, or solely in the negative domain of t, in spite of giving a mixture, may tend to make equation (4.1) inconsistent. Such a contingency will, of course, be very unlikely from the practical point of view.

VI. CONSISTENCY OF THE EQUATIONS RELATIVE TO THE ESTIMATION OF σ

The following observations on the relative magnitudes of the components of equation (4.2) which is relevant to the estimation of σ will indicate how the consistency of the equation may be affected by a small sample. Diagram II shows the graphs of tz/q and tz/p. It is easy to see that

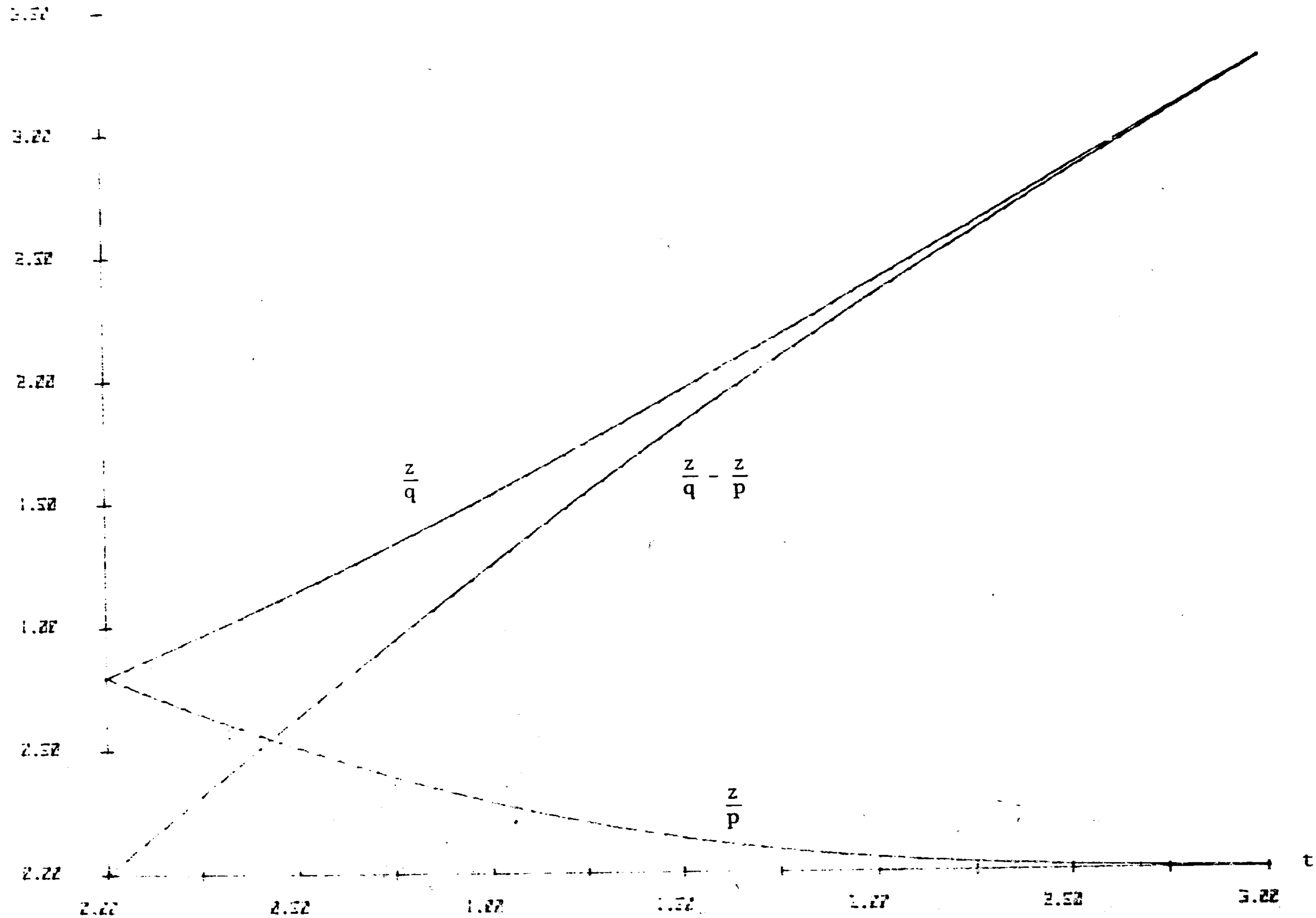

Diagram I. GRAPHS OF z/q, z/p, AND $(z/q - z/p)$

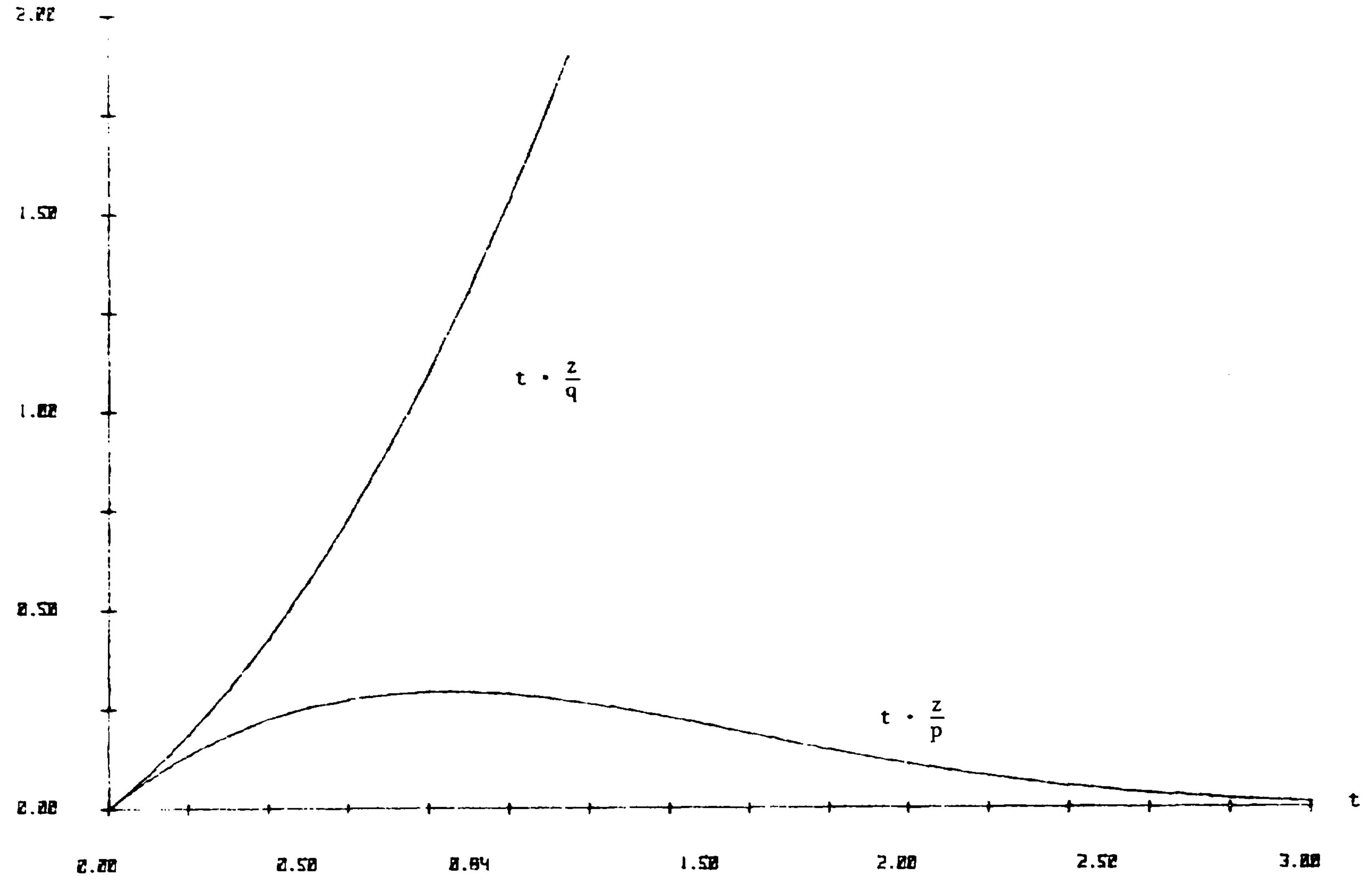

Diagram II. GRAPHS OF tz/q AND tz/p

$$\lim_{t\to\infty} \frac{z}{q} \to \infty, \qquad \lim_{t\to\infty} \frac{tz}{q} \to \infty,$$

$$\lim_{t\to\infty} \frac{z}{p} \to 0, \qquad \lim_{t\to\infty} \frac{tz}{p} \to 0.$$

The derivative of tz/p is given by

$$-\frac{z}{p}\left(t^2 + \frac{tz}{p} - 1\right). \tag{8}$$

At $t = 0$, the derivative (8) is positive being given by $(2/\pi)^{\frac{1}{2}}$. The function tz/p is increasing to the right of 0, and has a maximum at a point t satisfying $(t^2 + tz/p - 1) = 0$. The solution of the equation is obtained empirically at $t = 0.84$ where the value of the function tz/p is 0.29.

Similarly, the derivative of tz/q is also positive at $t = 0$ being given by $(2/\pi)^{\frac{1}{2}}$. This function goes to ∞ with t, and does not appear to have a maximum.

The value of tz/q at $t = 0.84$ is 1.18, while at this point, $tz/p = 0.29$, being the maximum in this neighborhood. On the right-hand side of $t = 0.84$, the value of tz/p goes on decreasing while that of tz/q goes on increasing. It will thus be evident from an inspection of the graphs in Diagram II that if observations are taken solely in the positive domain, such observations cannot be taken very far away from $t = 0.84$ to the right to ensure that equation (4.2) be consistent. Also, a failure has to be compensated by more than one success. With an interchange of p and q, a similar situation will emerge if observations are taken solely in the negative domain of t.

Thus, Diagrams I and II jointly reveal that, whether in the positive or in the negative domain, observations cannot be taken too far away from zero.

VII. ON THE DIVERGENCE (z/q - z/p)

Along with the graphs of z/q and z/p, the graph of (z/q - z/p), as mentioned in section V, is also shown in Diagram I, as this difference was vital in checking the consistency of equation (4.1) in a small sample. This divergence serves yet one more useful purpose for which the following algebraic development is necessary.

It will be noticed that $(z/q - z/p) \to \infty$, meeting z/q asymptotically as $t \to \infty$. The graph of (z/q - z/p) might give the impression as if it is a straight line. It is not so. The slope of the curve changes very slowly.

Let $C(t) = (z/q - z/p)$. Its first derivative is given by

$$C'(t) = (\frac{z}{q})^2 + (\frac{z}{p})^2 - \frac{zt}{q} + \frac{zt}{p} . \qquad (9)$$

It is shown in Appendix (A) that $C'(t) > 0$, indicating that $C(t)$ is increasing to the right of the origin. The value of (9) at $t = 0$ is positive being given by $4/\pi$.

It can also be easily shown that

$$\lim_{t \to 0} \frac{C(t)}{t} \to \frac{4}{\pi} = 1.2733 \quad . \qquad (10)$$

The equation of the tangent to $C(t)$ at $t = 0$ can therefore be written as

$$C(t) = \tan \theta \; t , \qquad (11)$$

where $\tan \theta = 1.2733$, with $\theta = 51.85°$.

We show in Appendix (B) (writing $B(t)$ for $C(t)/t$) that

$$\lim_{t \to \infty} \frac{C(t)}{t} = 1 \quad . \qquad (12)$$

Again, we show in Appendix (C) (writing $tB(t)$ for $C(t)$) that

$$\lim_{t \to \infty} C'(t) = 1.$$

Hence, we may say that the tangent at $t \to \infty$ makes an angle of 45° with the t-axis and that equation (12) represents the limiting form of the tangent at infinity.

It is thus observed that the slope starts with an angle of 51.85° at 0, and attains an angle of 45° at infinity. What happens in between is indicated below.

The second derivative of C(t) is given by

$$C''(t) = 2\left(\frac{z^3}{q^3} - \frac{z^3}{p^3} \right) - \left(\frac{z}{q} - \frac{z}{p} \right)$$

$$- 3t\left(\frac{z^2}{q^2} + \frac{z^2}{p^2} \right) + t^2\left(\frac{z}{q} - \frac{z}{p} \right). \tag{13}$$

$C''(t)$ is 0 at $t = 0$, and then becomes negative until it goes to zero again at $t = 3.0731$, indicating that it is a point of inflexion of C(t). The values of C(t) and C'(t) at $t = 3.0731$ are obtained respectively as 3.3476 and 0.9426. Thus, the equation of the tangent to C(t) at (3.0731, 3.3476) may be obtained as

$$\frac{z}{q} - \frac{z}{p} = 0.9426t + 0.4509. \tag{14}$$

We can find the equations of the tangents also at other points. Such equations will be helpful in checking the solutions of some equations for which closed form solutions are not available (see section VIII).

The graph of $C''(t)$ has some interesting features. It is zero at $t = 0$, and then it becomes negative and remains so until it becomes zero again at $t = 3.0731$. From this point on, it remains positive, going to zero again in the neighborhood of $t = 6.287$. Although we do not need to know about its behavior beyond $t = 3.0731$ for the present purpose, the nature of the graph beyond this point may be interesting from the mathematical point of view.

VIII. ON THE EFFICIENCY OF ESTIMATION WHEN THE EQUATIONS ARE CONSISTENT

A. The Basis for the Estimation of Efficiency

The elements of the information matrix A, given in (7.1) - (7.3), are rewritten below as;

$$A_{\mu\mu} = \frac{1}{\sigma^2} \sum_i \frac{z_i^2}{p_i q_i} = \frac{1}{\sigma^2} \sum_i A(t_i), \tag{15.1}$$

$$A_{\mu\sigma} = \frac{1}{\sigma^2} \sum_i \frac{z_i^2 t_i}{p_i q_i} = \frac{1}{\sigma^2} \sum_i A(t_i) t_i, \tag{15.2}$$

$$A_{\sigma\sigma} = \frac{1}{\sigma^2} \sum \frac{z_i^2 t_i^2}{p_i q_i} = \frac{1}{\sigma^2} \sum_i A(t_i) t_i^2, \tag{15.3}$$

where $A(t_i)$ is defined to be $z_i^2/p_i q_i$.

In a large sample, we would expect $A_{\mu\sigma}$ to be 0. Since $A(t_i)$ is positive, and that $A(t_i) = A(-t_i)$, $A_{\mu\sigma}$ would tend to 0 with a large set of $\pm$ t. A small sample also could be taken, if observations could be controlled, in such a manner as to make $A_{\mu\sigma}$ zero, or nearly zero. In fact, we would take the observations, if we could, at such values of t as would make $A_{\mu\sigma}$ zero, and, at the same time, make det. (A) maximum, or det. (A^{-1}) minimum, in order that the efficiency of estimation be maximized. This desideratum of D-optimality is in keeping with the criterion as is sometimes adopted in Design of Experiments to define the overall efficiency of an experiment. Such maximization of det (A) or minimization of det. (A^{-1}) would bring about an overall reduction of variances of the estimates, although we would have liked that the variance of each of the estimates be minimized individually, if possible.

Det. (A) is given by

$$\text{Det. (A)} = [A_{\mu\mu} A_{\sigma\sigma} - (A_{\mu\sigma})^2] \quad . \tag{16}$$

Expression (16) will be maximized if $(A_{\mu\mu} A_{\sigma\sigma})$ is maximized, and $(A_{\mu\sigma})^2$ is minimized. Since $(A_{\mu\sigma})^2$ is non-negative, the minimum possible value would be given by 0, and this would be consistent with the requirement that the estimates of μ and σ^2 be uncorrelated. Before the problem of maximizing $(A_{\mu\mu} A_{\sigma\sigma})$ is taken up, it may be appropriate to examine the problem of maximizing each of $A_{\mu\mu}$ and $A_{\sigma\sigma}$ separately.

If it was desired to estimate only μ, we would have utilized only one equation, equation (4.1), for this purpose, and, in that case, the variance of $\hat{\mu}$ would have been given by $(A_{\mu\mu})^{-1}$. Similarly, if it was required to estimate only σ, we would have made use of equation (4.2), and the variance of $\hat{\sigma}$ would have been given by $(A_{\sigma\sigma})^{-1}$. Therefore, in these two separate problems, we would have maximized either $A_{\mu\mu}$ or $A_{\sigma\sigma}$.

B. Maximum of $A_{\mu\mu}$

Since $A_{\mu\mu} = \sum_i A(t_i)/\sigma^2$, and each $A(t_i)$ is positive, the maximum of $A_{\mu\mu}$ will be attained if each $A(t_i)$ is maximized. This condition will be consistent with the admissibility of replications at a t_i. $A(t)$ is given by

$$A(t) = \frac{z^2}{pq} \quad . \tag{17}$$

The graph of (17) as a function of t in the positive domain of t is given in Diagram III. The graph shows that (17) has a maximum at $t = 0$. Differentiating (17) with respect to t, we obtain

$$A'(t) = -\frac{z^2}{pq}\left(2t + \frac{z}{p} - \frac{z}{q}\right). \tag{18}$$

An obvious solution of the equation $2t + z/p - z/q = 0$ is given by $t = 0$, and, at this point, the value of $A(t)$ is $2/\pi$. In view of what has been noted in section VII, the equation (18) cannot have any other solution since $(z/q - z/p)$ can hardly be as large as $2t$. The maximum possible value of $A_{\mu\mu}$, admitting repetitions at the same value of t, will be given by

$$\frac{2n}{\pi\sigma^2} \quad , \tag{19}$$

where n is the size of the sample. Hence, the minimum possible variance for $\hat{\mu}$ will be given by

$$\frac{\pi\sigma^2}{2n} \quad . \tag{20}$$

It may be difficult to attain this ideal minimum limit in actual practice. But, observations close to $t = 0$, that is, close to the mean, and approaching the mean from both sides, would bring down the variance close to it. We may approach the mean from both sides because the value of (17) remains the same for $\pm$ t.

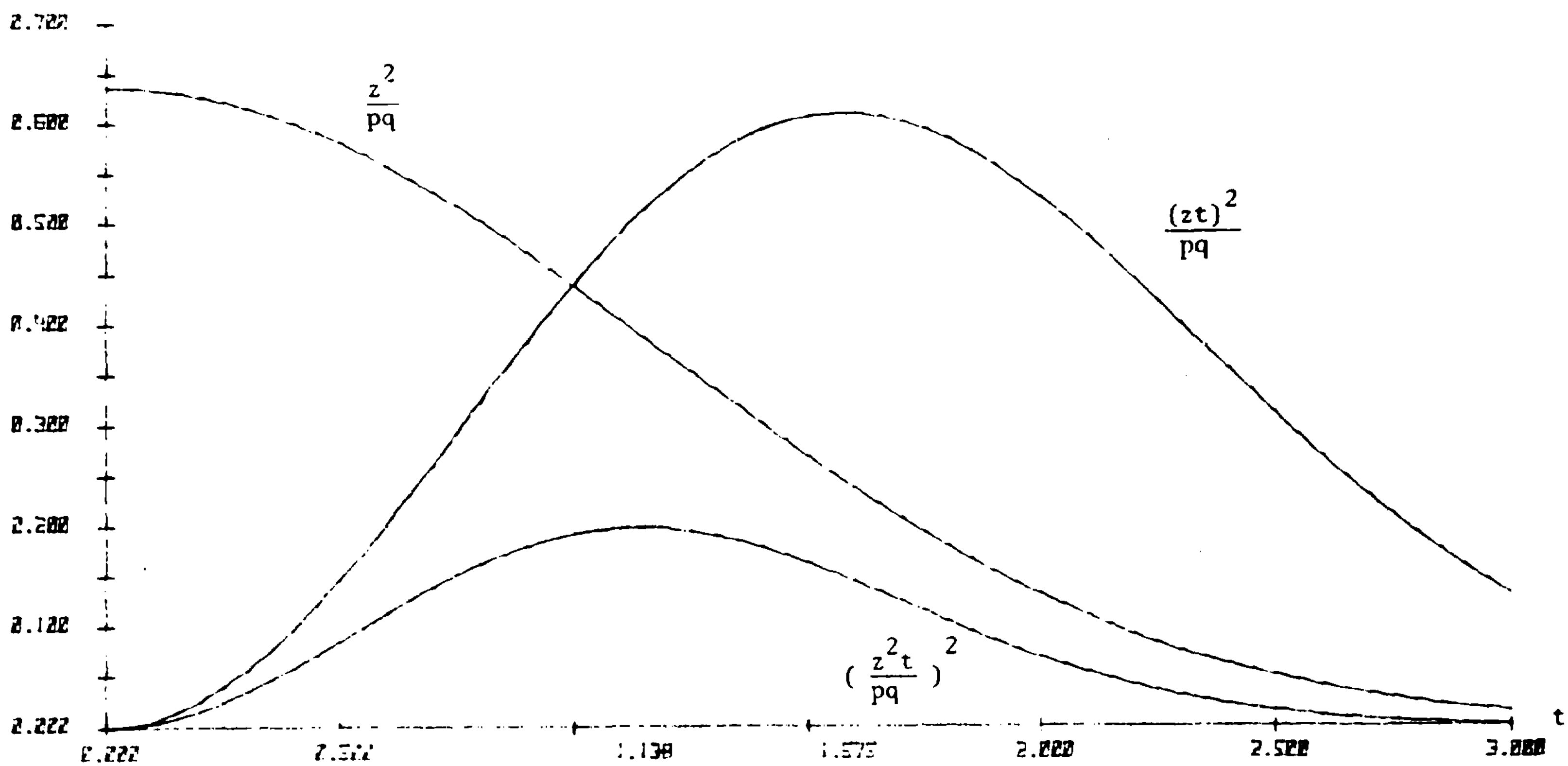

Diagram III. GRAPHS OF z^2/pq, $(zt)^2/pq$, AND $(z^2t)^2/(pq)^2$

C. Maximum of $A_{\sigma\sigma}$

Since $A_{\sigma\sigma} = \sum_i A(t_i)t_i^2/\sigma^2$, and each of the constituent terms is positive, $A_{\sigma\sigma}$ will be maximized, if each $(A_i t_i^2)$ is maximized. We have

$$A(t)t^2 = \frac{z^2 t^2}{pq} \quad . \tag{21}$$

The graph of (21) as a function of t in the positive domain of t is indicated in Diagram (III). It would show that (21) has a maximum. The derivative of $(A(t)t^2)$ is given by

$$(A(t)t^2)' = -\frac{z^2 t}{pq}\left[2t^2 - t\left(\frac{z}{q} - \frac{z}{p}\right) - 2\right] \quad . \tag{22}$$

The maximum of (21) will be obtained at a point t given by the solution of the equation,

$$2t^2 - t\left(\frac{z}{q} - \frac{z}{p}\right) - 2 = 0 \quad . \tag{23}$$

The solution t = 0 coming from (22) is ruled out. The zero of the function in the positive domain, found empirically, is given by t = 1.5750, and the value of $(A(t)t^2)$ at this point is given by $1.9114/\pi$. Hence, the maximum possible value of $A_{\sigma\sigma}$ will be given by

$$\frac{n(1.9114)}{\pi\sigma^2} , \tag{24}$$

where n is the size of the sample. Thus, the minimum possible variance of $\hat{\sigma}$ may be obtained as

$$\frac{\pi\sigma^2}{n(1.9114)} \cdot \tag{25}$$

Here also, observations close to $t = \pm\ 1.5750$, that is, close to $+\ 1.5750\sigma$ and $-\ 1.5750\sigma$ from the mean, would bring down the variance close to what is shown in (25). The minus sign of t is admitted because the value of (21) remains the same for $\pm$ t.

The solution of equation (23), as mentioned above, has been obtained empirically. However, the solution may be checked with the help of the results indicated in section VII. The equation of the tangent to $C(t)$ at a point $(t_0, C(t_0))$ may be obtained as $C(t) = C'(t_0)t + C(t_0) - C'(t_0)t_0$. In this particular instance, $t_0 = 1.5750$, $C(t_0) = 1.8802$, and $C'(t_0) = 1.0644$. Hence, for $(z/q - z/p)$ in equation (23), we may substitute $(1.0644t + .2037)$. With this substitution, the equation (23) will reduce to

$$t^2 - .2177t - 2.1377 = 0 \,. \qquad (26)$$

The positive root of equation (26) is obtained as 1.5750. This root checks the solution obtained empirically.

D. <u>Maximum of $(A_{\mu\mu}A_{\sigma\sigma})$</u>

In the joint estimation of μ and σ, we would be interested in maximizing the product,

$$(A_{\mu\mu}A_{\sigma\sigma}) = \frac{1}{\sigma^4}(\sum_i A(t_i))(\sum_i A(t_i)t_i^2) \qquad (27)$$

as $(A_{\mu\sigma})^2$ would be expected to go to zero. One way would be to take the product of the maxima of $A_{\mu\mu}$ and $A_{\sigma\sigma}$ as the required maximum of $(A_{\mu\mu}A_{\sigma\sigma})$, being given by

$$\frac{2n^2(1.9114)}{\pi^2\sigma^4} \,, \qquad (28)$$

where n is the sample size. It will hardly be possible to reach this upper limit in a problem of jointly estimating both μ and σ. Alternatively, a less ambitious maximum may be provided by n^2 times the maximum of the product,

$$[A(t) \times A(t)t^2] = [A(t)t]^2. \qquad (29)$$

The graph of (29) as a function of t in the positive domain is indicated in Diagram (III). The function appears to have a maximum. The derivative of $(A(t)t)^2$ is given by

$$[(A(t)t)^2]' = \frac{-2z^4t}{(pq)^2}\,[2t^2 - t(\frac{z}{q} - \frac{z}{p}) - 1] \,. \qquad (30)$$

In the context of finding the maximum, the root $t = 0$ is ruled out. The maximum will be given by the positive root of the equation

$$2t^2 - t(\frac{z}{q} - \frac{z}{p}) - 1 = 0. \tag{31}$$

The positive root is empirically obtained at $t = 1.1381$. Hence, the observations have to be taken close to $\pm(1.1381\sigma)$ from the mean. As in the two other cases, the minus sign is taken as admissible, because (29) remains the same for $\pm$ t.

Here also, the solution may be checked by the same device as referred to earlier.

The value of A(t) at $t = 1.1381$ is given by

$$\frac{1.2304}{\pi} = 0.3916 \quad , \tag{32}$$

and the value of $(A(t)t^2)$ at $t = 1.1381$ is given by

$$\frac{1.5937}{\pi} = 0.5073 \quad . \tag{33}$$

Hence, the maximum of $(A_{\mu\mu}A_{\sigma\sigma})$ by the maximum of the product in (29) is given by

$$\begin{aligned} &[\ \frac{n(1.2304)}{\pi\sigma^2}\ ] \times [\ \frac{n(1.5937)}{\pi\sigma^2}\ ] \\ &= \frac{n^2}{\pi^2\sigma^4} (1.9609) \\ &= \frac{n^2}{\sigma^4} (.1987) \ . \end{aligned} \tag{34}$$

In this set-up, when $A_{\mu\sigma} = 0$, the minimum possible variances of $\hat{\mu}$ and $\hat{\sigma}$ would be given respectively by

$$\frac{\pi\sigma^2}{n(1.2304)} \text{ and } \frac{\pi\sigma^2}{n(1.5937)} , \tag{35}$$

that is, by

$$\frac{\sigma^2}{n}(2.5530) \text{ and } \frac{\sigma^2}{n}(1.9712) \ . \qquad (36)$$

E. Recommendation

Hence, in the problem of joint estimation of μ and σ with maximized efficiency for both, the recommended procedure would be to aim at taking the observations in the neighborhood of $t = \pm\ 1.1381$, that is, at a distance of $(\pm\ 1.1381)\sigma$ from the mean. If we take the same number of observations on either side of the mean at the same distance, $A_{\mu\sigma}$ will be zero.

In this context, it may be pertinent to observe that the point $t = 0$ makes $A(t)$ the maximum, but contributes nothing to $A(t)t^2$. On the other hand, the points at $t = \pm\ 1.5750$ make $A(t)t^2$ the maximum, while adding something positive, although not very large, to $A(t)$.

In spite of the fact that observations in the neighborhood of $t = 0$ would result in increased efficiency in the estimation of μ, while not being contributory at all to the estimation of σ, aiming at $t = 0$ might help in getting a zone of mixed results.

IX. ASSESSMENT OF EFFICIENCY OF A RECORDED EXPERIMENT

Golub and Grubbs[1] used this methodology to provide the estimates of μ and σ, as well as the estimates of σ_{μ}^{2} and σ_{σ}^{2} for the following experiment.

In firing five rounds of a given projectile of a given armor plate, the following observations were recorded:

Velocity (f/s)	Condition of Impact
2433	Non-penetration
2415	Non-penetration
2415	Non-penetration
2453	Penetration
2423	Penetration

In the above example, the sample size n = 5. The estimates of $\hat{\mu}$ and $\hat{\sigma}$ were obtained[1] respectively as 2431.6 f/s and 15.0 f/s, and of $\sigma^2_{\hat{\mu}}$ and $\sigma^2_{\hat{\sigma}}$ as 115.5 and 155.5, respectively. The variance-covariance matrix was obtained as

$$A^{-1} = \begin{bmatrix} .01011 & -.00330 \\ -.00330 & .00751 \end{bmatrix}^{-1} = \begin{bmatrix} 115.5 & 50.8 \\ 50.8 & 155.5 \end{bmatrix} .$$

The value of det (A) = .000065036, while the value of (34), substituting $\hat{\sigma}^2$ for σ^2, is .00009812. Hence, the ratio of the two is .6635 which means that the efficiency of estimation in this particular experiment has been obtained as 66.35%. (It may be mentioned here that the efficiency could also have been calculated by dividing $\sigma^4/n^2(.1987)$ by det (A^{-1}) to get whole numbers, rather than fractions, in the division.)

The ideal variance of $\hat{\mu}$, by (36), would have been 114.89, while, in the experiment, $\sigma^2_{\hat{\mu}}$ has been obtained as 115.5. Thus, the efficiency in the estimation of μ is obtained as 114.89/115.50 = 99.47% which is very high.

The ideal variance of $\hat{\sigma}$, by (36), would have been obtained as 88.70, while, in this particular experiment, $\sigma^2_{\hat{\sigma}}$ has been obtained as 155.5. Thus, the efficiency in estimating $\hat{\sigma}$ may be calculated as 88.70/155.5 = 57.04%.

Now, let us examine the values of t of this particular experiment which are given below:

R	$\hat{t}$
2415	-1.11
2415	-1.11
2423	-0.57
2433	0.09
2453	1.43

*One may find slight arithmetical discrepancies in these estimates, but such small discrepancies will not be of any material consequence.

The values of t in the neighborhood of $t = 0$ have been responsible to render the variance of $\hat{\mu}$ almost ideally small. If there were a few more observations in the neighborhood of $t = \pm 1.57$, the variances of $\hat{\sigma}$ would have been almost ideally smaller.

An observation at $t = 1.43$ has helped in the reduction of $\sigma^2_{\hat{\sigma}}$ to some extent.

If the estimated variances of $\hat{\mu}$ and $\hat{\sigma}$ as obtained in this experiment would have compared with the more ambitious limits of variances as given in (20) and (25), the calculated efficiency of this experiment would be less.

X. REMARK

It appears that many experiments have, in the past, been analyzed by this procedure and we have, with us, the estimates of μ, σ, and $\sigma^2_{\hat{\mu}}$, $\sigma^2_{\hat{\sigma}}$. It may be worthwhile to calculate the efficiencies of those experiments in the light of this discussion to let the experimenters know where they stand in retrospect with regard to the estimates of μ and σ in each case. This search may be of help in the matter of perspective planning.

XI. ACKNOWLEDGMENT

We are grateful to Mr. Miguel Andriolo of BRL for some of the numerical calculations, and to Mr. W. Egerland of BRL for the Appendices.

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APPENDIX A*

We wish to show that the first derivative of

$$C(t) = z\left(\frac{1}{q} - \frac{1}{p} \right)$$

is positive.

We have

$$C'(t) = -\, tz\left(\frac{1}{q} - \frac{1}{p} \right) + z^2\left(\frac{1}{q^2} + \frac{1}{p^2} \right)$$

$$= \frac{z}{q} \left[\frac{z - qt}{q} \right] + \frac{z^2}{p^2} + t\,\frac{z}{p} .$$

The function $z - qt$ is positive at $t = 0$, has a negative derivative, and tends to zero as $t \to \infty$. Therefore, $z - qt \geq 0$ for $t \geq 0$.

Hence, $(tB(t))' > 0$.

*Appendices A, B and C are due to Mr. W. Egerland.

APPENDIX B

Consider the function,

$$B(t) = t^{-1} \left(\frac{z(t)}{q(t)} - \frac{z(t)}{p(t)} \right) .$$

We wish to show that $\lim_{t \to \infty} B(t) = 1$.

We note first, since $\lim_{t \to \infty} p(t) = 1$, that

$$\lim_{t \to \infty} B(t) = \lim_{t \to \infty} \frac{z(t)}{tq(t)} . \qquad (*)$$

Since the limits of both $z(t)$ and $tq(t)$ are zero as $t \to \infty$, we may apply l'Hospital's rule for the evaluation of (*) to obtain

$$\lim_{t \to \infty} B(t) = \lim_{t \to \infty} \frac{z'(t)}{q(t) - tz(t)} \qquad (**)$$

The right hand side of (**) is again of the form [0/0], so we may apply the rule again. Using $z'(t) = -tz(t)$, $z''(t) = (t^2 - 1)z(t)$, we have

$$\lim_{t \to \infty} B(t) = \lim_{t \to \infty} \frac{z''}{-2z(t) + t^2 z(t)}$$

$$= \lim_{t \to \infty} \frac{t^2 - 1}{t^2 - 2}$$

$$= 1 .$$

APPENDIX C

We wish to show that $\lim_{t\to\infty} (tB(t))' = 1$.

Since $(tB(t))' = B(t) + tB'(t) = B(t) + tB(t)\,\frac{d}{dt}\log B(t)$, it is sufficient to show that $tB(t)\frac{d}{dt}\log B(t) \to 0$ as $t \to \infty$. Writing $B(t)$ in the form

$$B(t) = \frac{z(t)(2p(t)-1)}{tp(t)q(t)} \quad ,$$

it follows after some manipulation that

$$tB(t)\,\frac{d}{dt}\log B(t) = t\left\{ \frac{z'}{z} + \frac{2p'}{2p-1} - \frac{p'}{p} - \frac{q'}{q} - \frac{1}{t} \right\} B(t)$$

$$= \left[\frac{tz}{p(2p-1)} + \left\{ \frac{-t^2q + tz - q}{q} \right\}\right] B(t)$$

$$= \left[\frac{tz}{p(2p-1)} + \beta(t)\right] B(t).$$

We can omit $B(t)$, since $B(t) \to 1$, as $t \to \infty$.

Remembering that t^2q and tz go to 0 as $t \to \infty$, we have, by l'Hospital's rule,

$$\lim_{t\to\infty} \beta(t) = \lim_{t\to\infty} 2\left(1 + \frac{tq}{z} \right).$$

But, the limit on the right is zero, since from Appendix B,

$$\lim_{t\to\infty} \frac{tq}{z} = \lim_{t\to\infty} \frac{1}{B(t)} = 1.$$

Hence, $\lim_{t\to\infty} (tB(t))' = 1$.

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